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SOME DIFFERENTIAL INVARIANTS OF SUBMANIFOLDS OF EUCLIDEAN SPACE

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1. Introduction

Let $f: M^s \to E^n$ be a C^∞ immersion of an oriented differentiable manifold with or without boundary into Euclidean space of dimension n, p an arbitrary generic (in a sense which will be made clear in § 2) point of E^n, and N a fiber space over M which is mapped in a C^∞ fashion by a function g into E^n. In this paper we prove a number of differential topological and integral geometric formulas relating the intersection number of N with p to the integrals of certain differential invariants of M.

In § 2, we prove the main equation from which all our results follow. In § 3, we consider the case where $f: M^{n-1} \to E^n$ is an immersion of a hypersurface and N is a particular submanifold of the normal bundle. Here the intersection number is seen to relate the normal degree of the immersion to the linking number of the immersion with the point p.

In § 4, we consider the simple case of curves in three-space and find new integral formulas for the total curvature and total torsion of a closed space curve. In § 5, we present the general theory in which we introduce, for s odd, integral formulas for new differential invariants generalizing the curvature and torsion of a space curve. For s even we obtain differential topological results relating the Euler classes of certain s-plane bundles to our intersection number. In particular, in § 6 we prove that if $f: M^s \to E^{n=s+k}$ is an immersion of an oriented compact manifold M^s and if N is a k-plane bundle over M^s and p a point of E^n, then the intersection number of N with p is the Euler class of the complementary s-plane bundle evaluated on the fundamental class of M^s.

Finally, § 7 deal with manifolds M^s with boundary and gives a new formulation of the Gauss-Bonnet theorem for arbitrary codimension.

In all that follows all manifolds and fiber spaces are to be assumed C^∞ and oriented, and all maps are to be asumed C^3.

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2. The main equation

Let N^n be a compact orientable differentiable manifold with boundary ∂N, $g\colon N \to E^n$ a differentiable map, and p a point of E^n such that $g(\partial N)$ does not intersect p. We denote by $p \times N$ the usual cartesian product of p with N, i.e., $\{(p, n) \mid n \in N\}$, and let I denote the intersection locus of $g(N)$ and p, i.e.,

$$I = \{(p, n) \in p \times N \mid p = g(n)\} .$$

By means of the Thom Transversality theorem it can be shown that, under a small deformation of g, these intersections may be made transverse. This may be done without loss of generality since the geometric entities we shall be discussing vary continuously. Hence, we shall assume the intersections transverse. Because of the compactness of $g(N)$, they will be finite in number, say r; denote them by

$$(p, n^{(a)}) , \qquad a = 1, \cdots, r .$$

We surround each of these points of I by small disjoint discs, D_a.

We next define a map

$$e\colon p \times N - I \to S^{n-1} ,$$

where S^{n-1} is the unit $(n-1)$-sphere in E^n. For each $(p, n) \in p \times N - I$, we set

$$e(p, n) = \frac{g(n) - p}{|g(n) - p|} .$$

Let $d0_{n-1}$ be the pull-back under e^* of the volume element of S^{n-1}.

We orient $p \times N$ by means of the orientation on N; this induces orientations on the discs D_a and hence on ∂D_a, $a = 1, \cdots, r$. We will take the orientation on ∂D_a to be that given from the "inside." Note that $d(d0_{n-1}) = 0$ (where d denotes the exterior derivative) since $d0_{n-1}$ is a $(n-1)$-form on an $(n-1)$-dimensional manifold.

Applying Stokes' theorem, we obtain

$$0 = \int_{p \times \partial N} d0_{n-1} - \sum_{a=1}^{r} \int_{\partial D_a} d0_{n-1} .$$

The sum is simply $0_{n-1} I(g, p)$ where 0_{n-1} is the volume of the $(n-1)$-sphere and $I(g, p)$ is the algebraic number of intersections of $g(N)$ with p, or the sum of the indices of the intersections of $g(N)$ with p. For details, refer to [4]. This gives us, then, our main equation

$$\text{(E)} \qquad \frac{1}{0_{n-1}} \int_{p \times \partial N} d0_{n-1} = I(g, p) .$$

In what follows we apply this equation to various cases depending on our choices for N and g.

3. $N = M^{n-1} \times L,\ L = [a, b]$

For our first example we choose for N the cartesian product of an $(n-1)$-dimensional oriented compact differentiable manifold M without boundary with a closed interval $L = [a, b]$ of real numbers. Let $f\colon M^{n-1} \to E^n$ be a differentiable immersion of M into E^n, and suppose that v is a nonvanishing differentiable unit vector field on M. Then, we define the map g for equation (E) as follows:

$$g(n) = g(m, l) = f(m) + l v_{f(m)} ,$$

$n = (m, l) \in M \times L$. Equation (E) yields

$$\frac{1}{0_{n-1}} \int_{p \times M_b} d0_{n-1} - \frac{1}{0_{n-1}} \int_{p \times M_a} d0_{n-1} = (-1)^{n-1} I(g, p) ,$$

where we have denoted $M \times \{a\}$ and $M \times \{b\}$ respectively by M_a, M_b. The factor $(-1)^{n-1}$ comes from the induced orientation on ∂N. This equation gives

$$L(p, M_b) - L(p, M_a) = (-1)^{n-1} I(g, p) ,$$

Where $L(p, M_i)$ is the linking number of the point p with the immersed manifold M moved a distance i along v.

A most interesting situation arises when $b = \infty$ and $a = 0$, i.e., when $L = [0, \infty]$. Note that N is still compact as we add to the half-line L the point at ∞. For a discussion of this point see [4]. Let us examine the integral

$$\frac{1}{0_{n-1}} \int_{p \times M_\infty} d0_{n-1} .$$

Consider a point at ∞ which arises from the half line L through the point $f(m)$ along $v_{f(m)}$. Then the vector $e(p, m, \infty)$ is easily seen to be the unit vector $v_{f(m)}$ translated to the origin, for the line joining p to a point at ∞ must be parallel to the line along $v_{f(m)}$ from $f(m)$ to the same point at ∞. Thus, we can write

$$\frac{1}{0_{n-1}} \int_{p \times M_\infty} d0_{n-1}$$

as an integral over M in the following manner. Let $f(m)e_1, \cdots, e_n$ be an orthonormal frame such that e_1 is along $v_{f(m)} = e(p, m, \infty)$ and such that the orientation agrees with that of N. If we set $\omega_{ij} = de_i \cdot e_j$, then

$$\frac{1}{0_{n-1}} \int_{p \times M_\infty} d0_{n-1} = \frac{1}{0_{n-1}} \int_M \omega_{12} \wedge \cdots \wedge \omega_{1n} .$$

Thus, in the case $L = [0, \infty]$, equation (E) becomes

$$\frac{1}{0_{n-1}} \int_M \omega_{12} \wedge \cdots \wedge \omega_{1n} - L(p, M) = (-1)^{n-1} I(g, p) .$$

The most interesting case occurs when v is the unit normal vector field on M. Then, the first integral is simply the normal degree of the immersion, and $I(g, p)$ becomes the algebraic number of the intersections of the half-normal lines with the point p. We may call these intersections *normal intersections.*

Theorem 1. *Let $f\colon M^{n-1} \to E^n$ be an immersion of a compact orientable differentiable manifold without boundary. Then for a point $p \in E^n$, the*

$$\textit{normal degree} - L(p, M) = (-1)^{n-1} I(g, p) ,$$

where $L(p, M)$ is the linking number of p with M and $I(g, p)$ is the algebraic number of normal intersections.

Corollary 2. *If $n = 2$, then*

$$\frac{1}{2\pi} \int_M kds - L(p, M) = - I(g, p) ,$$

where k is the curvature of M.

Corollary 3. *If n is odd, then*

$$\chi(M) - 2L(p, M) = +2I(g, p) .$$

Corollary 3 follows from the Hopf theorem that the normal degree is one-half the Euler characteristic of M. Corollary 2 relates the linking or winding number of the curve about a point to the index of rotation.

4. Curves in E^3

We next discuss the situation in higher codimension, introducing it by considering first the simplest case, a curve in E^3. Let $f\colon M^1 \to E^3$ be an immersion of a curve in E^3 such that the curvature k never vanishes, say $k > 0$. An oriented two-plane bundle $\bar{N}$ over M consists of pairs (m, e) where $m \in M$ and e is a point in the two-plane through $f(m)$. For our purposes we "compactify" $\bar{N}$ by adding to each two-plane the oriented circle of points at infinity. Let v_1 and v_2 be unit orthonormal vector fields on M. Then we choose

$\bar{N}$ to be the two-plane bundle (oriented so that $v_1 v_2$ orients each plane) whose fiber at each point is the two-plane spanned by the vector v_1 and v_2. Recall that to use equation (E) we must have an N and a g. For N we choose a submanifold of $\bar{N}$ with boundary. At each point $m \in M$ we choose as the "fiber" for N the half-plane spanned by the line L along v_1 and the half-line along v_2 in its positive sense. If h is the map $h\colon \bar{N} \to E^3$ such that $h(m, e) = e$, we define the map g to be the restriction of h to N. Thus, if p is a point of E^3 such that $g(\partial N)$ does not intersect p, equation (E) yields

$$\frac{1}{4\pi} \int_{p \times \partial N} d\,0_2 = I(g, p) \, .$$

The ∂N consists of two parts, the portion at infinity, denoted $L_\infty(M)$, and the rest, denoted $L(M)$. Note that $g(L(M))$ is just the reuled surface swept out by the lines L. Our equation then becomes

$$-\frac{1}{4\pi} \int_{p \times L(M)} d\,0_2 + \frac{1}{4\pi} \int_{p \times L_\infty(M)} d\,0_2 = +I(g, p) \, .$$

The first integral is a Gauss integral and measures in some sense the linking of p with $L(M)$. We must keep in mind, however, that it is not necessarily an integer.

The second integral bears an analysis analogous to that of the first section. Let $f(m)$ be the image of a point of M and suppose l_∞ denotes a point of $L_\infty(M)$, say a point in the half-plane through $f(m)$. Then, as before, it is clear

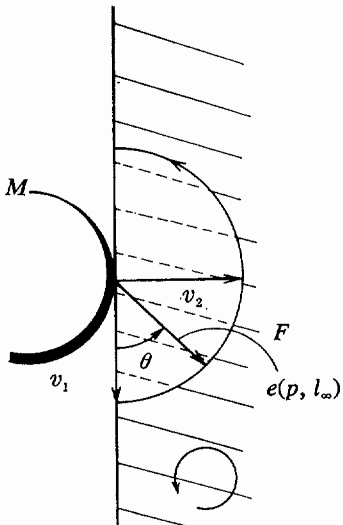

Fig. 1

that $e(p, l_\infty)$ is the same as a unit vector (translated to the origin) along the line from $f(m)$ to l_∞ in the half-plane through $f(m)$. Thus, as l_∞ takes on values from the half-circle of points at infinity in the half-plane through $f(m)$, $e(p, l_\infty)$ assumes all directions in the half-plane. Hence, the integral

$$\int_{p \times L_\infty(M)} dO_2$$

can be written as an integral over a fiber space over M, the fiber F being the half-circle of oriented directions in the half-plane. Cf. Figure 1.

To do this, we choose orthonormal frames e_1, e_2, e_3 such that e_1 is along $e(p, l_\infty)$, and e_2 is in the plane in which the line $\overline{pl_\infty}$ lies with the property that e_1e_2 agrees with the orientation $v_{1_{f(m)}}v_{2_{f(m)}}$ and such that $e_1e_2e_3$ is consistent with the orientation of $\bar{N}$, say $e_3 = e_1 \times e_2$. If we set $de_i \cdot e_j = \omega_{ij}$, we may write

$$\int_{p \times L_\infty(M)} dO_2 = \int_M \int_F \omega_{12} \wedge \omega_{13} ,$$

where F is the fiber mentioned above.

In order to integrate over the fiber, we use the techniques of [4]. We choose local fixed fields of orthonormal frames $f(m)a_1a_2a_3$ over M such that a_1 is along $v_{1_{f(m)}}$, a_2 is along $v_{2_{f(m)}}$ and $a_3 = a_1 \times a_2$. We may choose θ as a fiber parameter and write

$$\begin{aligned} e_1 &= a_1 \cos\theta + a_2 \sin\theta , \\ e_2 &= -a_1 \sin\theta + a_2 \cos\theta , \\ e_3 &= a_3 . \end{aligned}$$

Then we obtain, denoting $\pi_{ij} = da_i \cdot a_j$,

$$\begin{aligned} \omega_{12} &= d\theta + \pi_{12} , \\ \omega_{13} &= \pi_{13} \cos\theta + \pi_{23} \sin\theta , \end{aligned}$$

where the π_{ij}'s are defined on the base manifold M, and

$$\omega_{12} \wedge \omega_{12} = d\theta \wedge (\pi_{13} \cos\theta + \pi_{23} \sin\theta) ,$$

as any form of degree greater than 2 in the π_{ij}'s must vanish. If we choose for the canonical orientation base coordinates first, fiber coordinates last, we have

$$\frac{1}{4\pi}\int_{p\times L_\infty(M)} d0_2 = -\frac{1}{4\pi}\int_M\int_0^\pi (\pi_{13}\cos\theta + \pi_{23}\sin\theta)d\theta$$

$$= -\frac{1}{2\pi}\int_M \pi_{23} .$$

The main interest in this analysis, of course, lies in the choices of v_1 and v_2. If we choose the unit tangent T and principal normal N vector fields for v_1 and v_2, then a_3 is the binormal vector field B, and we obtain

$$-\frac{1}{2\pi}\int_M \pi_{23} = -\frac{1}{2\pi}\int_M dN\cdot B = -\frac{1}{2\pi}\int_M \tau ds ,$$

where τ is the torsion of the curve M. If we choose N and B for v_1 and v_2, then $a_3 = T$ and

$$-\frac{1}{2\pi}\int_M \pi_{23} = -\frac{1}{2\pi}\int_M dB\cdot T = 0 .$$

If we choose B and T for v_1 and v_2, then $a_3 = N$ and

$$-\frac{1}{2\pi}\int_M \pi_{23} = -\frac{1}{2\pi}\int_M dT\cdot N = -\frac{1}{2\pi}\int_M kds .$$

If we choose B and N for v_1 and v_2, then $a_3 = -T$ and

$$-\frac{1}{2\pi}\int_M \pi_{23} = -\frac{1}{2\pi}\int_M dN\cdot(-T) = -\frac{1}{2\pi}\int_M kds ,$$

and so on for the other possibilities.

What we obtain are integral formulas involving the ruled surfaces swept out by the tangent, normal and binormal lines. If we denote these respectively $T(M)$, $N(M)$, $B(M)$, we have

Theorem 4.

(1) $$-\frac{1}{4\pi}\int_{p\times T(M)} d0_2 - \frac{1}{2\pi}\int_M \tau ds = I(g_1, p) ,$$

(2) $$-\frac{1}{4\pi}\int_{p\times N(M)} d0_2 = I(g_2, p) ,$$

(3) $$-\frac{1}{4\pi}\int_{p\times B(M)} d0_2 - \frac{1}{2\pi}\int_M kds = I(g_3, p) ,$$

where $I(g_1, p)$ is the intersection number of the half-planes spanned by the full tangent line and half-(principal normal or binormal) line, $I(g_2, p)$ is the intersection number of the half-planes spanned by the full normal line and half-(tangent or binormal) line, and $I(g_3, p)$ is the intersection number of the half-planes spanned by the full binormal line and half-(tangent or principal normal) line.

The second integral equation is interesting in a special way since

$$\frac{1}{4\pi}\int_{p\times N(M)} d\,0_2$$

is an integer, whereas

$$\frac{1}{4\pi}\int_{p\times T(M)} d\,0_2 \quad \text{and} \quad \frac{1}{4\pi}\int_{p\times B(M)} d\,0_2$$

assume a continuum of values.

5. The general theory

Let $f\colon M^s \to E^n$ be an immersion of an s-dimensional oriented compact manifold without boundary into Euclidean n-space. Let v be a unit non-vanishing differentiable vector field on M (not necessarily a tangent vector field), and $\bar{N}$ an oriented k-plane bundle on M, where $k = n - s$, such that each fiber contains the line generated by v, i.e., such that at each point $f(m)$ the k-plane contains the line generated by $v_{f(m)}$. As before, we include in $\bar{N}$ for each k-plane the oriented $(k - 1)$-sphere of points at infinity.

In order to apply equation (E) we need to choose our N and g. We do this as follows. In each fiber of $\bar{N}$, let $W_{f(m)}$ be the $(k - 1)$-plane orthogonal to $v_{f(m)}$ at $f(m)$. N, then, will be the fiber space over M such that at each point $f(m)$, the fiber is the half k-plane spanned by $W_{f(m)}$ and the *positive* or *forward* half-line along $v_{f(m)}$. N is thus an n-dimensional submanifold of $\bar{N}$ whose boundary is the $(k - 1)$-plane bundle with W as fiber plus the portion of ∂N at infinity; we denote the former by $W(M)$ and the latter by $W_\infty(M)$. As in the case of curves in three-space, the image of $W(M)$ in E^n may be thought of as the manifold swept out by the $(k - 1)$-planes $W_{f(m)}$, $m \in M$.

Now any point of $\bar{N}$ is of the form (m, e) where $m \in M$ and e is a point of the k-plane through $f(m)$. If h is the map defined by $h\colon \bar{N} \to E^n$ such that $h(m, e) = e$, we define our map g to be the restriction of h to N. Equation (E) now yields

$$-\frac{1}{0_{n-1}}\int_{p\times W(M)} d\,0_{n-1} + \frac{1}{0_{n-1}}\int_{p\times W_\infty(M)} d\,0_{n-1} = (-1)^{n-1}I(g, p)\ ,$$

where p is a point of E^n such that $g(\partial N)$ does not intersect p.

The first integral is again a kind of Gauss integral. The analysis of the second is similar to the previous sections. Let $f(m)$ be the image of a point of M and suppose w_∞ denotes a point of $W_\infty(M)$, say a point in the half-k-plane through $f(m)$. Then it is clear that the $e(p, w_\infty)$ is the same as a unit vector (translated to the origin) along the line from $f(m)$ to w_∞. Thus, as w_∞ takes on values from the half-$(k-1)$-sphere of points at infinity in the half k-plane through $f(m), e(p, w_\infty)$ assumes all directions in the half k-plane. Thus the integral

$$\int_{p \times W_\infty(M)} d0_{n-1}$$

is an integral over the fiber space of oriented directions in the half k-plane, the fiber F being the half $(k-1)$-sphere.

More explicitly, let us choose orthonormal frames $e_1, \cdots, e_n$ such that e_1 is along the map $e(p, w_\infty), e_2, \cdots, e_k$ lie in the k-plane in which the line $\overline{pw_\infty}$ or $\overline{f(m)w_\infty}$ lies such that $e_1, \cdots, e_k$ agrees with the orientation given the fiber of $\bar{N}$, and such that $e_{k+1}, \cdots, e_n$ are normal to the k-plane with the property that $e_1, \cdots, e_n$ agrees with the orientation of $\bar{N}$. Then, if we write $de_i \cdot e_j = \omega_{ij}$, we have

$$\frac{1}{0_{n-1}} \int_{p \times W_\infty(M)} d0_{n-1} = \frac{1}{0_{n-1}} \int_M \int_F \omega_{12} \wedge \cdots \wedge \omega_{1n} ,$$

where F is the fiber mentioned above.

In order to integrate over the fiber we choose local fixed fields of orthonormal frames $f(m)a_1, \cdots, a_n$ such that $a_1, \cdots, a_k$ span the k-plane with a_k along $v_{f(m)}$ and agree with the orientation, and such that $a_{k+1}, \cdots, a_n$ are normal to the k-plane, with $a_1 \cdots a_n$ agreeing with the orientation of $\bar{N}$. Then we may write

$$\begin{aligned} e_1 &= u_{11}a_1 + \cdots + u_{1k}a_k , \\ &\vdots \qquad\qquad\quad \vdots \\ e_k &= u_{k1}a_1 + \cdots + u_{kk}a_k , \qquad (u_{ij}) \text{ orthogonal} , \\ e_t &= a_t , \quad t = 1 + k, \cdots, n . \end{aligned}$$

Then

$$\begin{aligned} \omega_{12} \wedge \cdots \wedge \omega_{1n} &= \omega_{12} \wedge \cdots \wedge \omega_{1k} \wedge \omega_{1k+1} \wedge \cdots \wedge \omega_{1n} \\ &= (d0_{k-1} + \text{terms in } \pi_{ij}\text{'s}) \wedge u_{1i_1} \cdots u_{1i_s} \pi_{i_1 k+1} \wedge \cdots \wedge \pi_{i_s n} , \end{aligned}$$

where we have used the Einstein smmation convention for repeated indices, the range of the i_r being from 1 to k. Since the π_{ij}'s are defined on the base manifold M, we have that any form of degree $>s$ is identically zero. Hence,

$$\omega_{12} \wedge \cdots \wedge \omega_{1n} = d0_{k-1} \wedge u_{1i_1} \cdots u_{1i_s} \pi_{i_1 1+k} \wedge \cdots \wedge \pi_{i_s n} ,$$

$$\int_{p \times W_\infty(M)} d0_{n-1} = \int_M \int_F d0_{k-1} \wedge u_{1i_1} \cdots u_{1i_s} \pi_{i_1 1+k} \wedge \cdots \wedge \pi_{i_s n} .$$

Integrating, we obtain, for s *odd*,

$$\frac{1}{0_{n-1}} \int_{p \times W_\infty(M)} d0_{n-1} = -\frac{1}{0_s} \int_M k_v(N) dV ,$$

where dV is the volume element of M and $k_v(N)dV$ is defined as follows. Let

$$\Phi_l = \varepsilon_{\alpha_1 \cdots \alpha_s} \pi_{k\alpha_1} \wedge \cdots \wedge \pi_{k\alpha_{s-2l}} \wedge \Lambda_{\alpha_{s-2l+1}\alpha_{s-2l+2}} \wedge \cdots \wedge \Lambda_{\alpha_{s-1}\alpha_s} ,$$

where

$$\Lambda_{\alpha\beta} = \sum_{i=1}^{k-1} \pi_{\alpha i} \wedge \pi_{i\beta} ,$$

$$\varepsilon_{\alpha_1 \cdots \alpha_s} = \begin{cases} +1 \text{ if } \alpha_1 \cdots \alpha_s \text{ is an even permutation of } 1+k, \cdots, n , \\ -1 \text{ if } \alpha_1 \cdots \alpha_s \text{ is an even permutation of } 1+k, \cdots, n , \\ 0 \text{ otherwise.} \end{cases}$$

Define

$$\Lambda = \frac{1}{\pi^{\frac{1}{2}(s+1)}} \sum_{l=0}^{\frac{1}{2}(s-1)} (-1)^l \frac{1}{1 \cdot 3 \cdots (s-2l) 2^{\frac{1}{2}(s+1)+l} l!} \Phi_l .$$

Then

$$k_v(N) dV = (-1)^k 0_s \Lambda .$$

(The $(-1)^k$ is due to our choice of canonical orientation of base coordinates first.) We call $k_v(N)dV$ the *curvature form with respect to the vector field v and k-plane N*, for it depends on the choice of k-plane and vector field.

The situation for s even is much simpler. In this case, we obtain

$$\frac{1}{0_{n-1}} \int_{p \times W_\infty(M)} d0_{n-1} = \frac{(-1)^{s/2}}{2^{s+1}\pi^{s/2}(s/2)!} \int_M \Delta_0 ,$$

where $\Delta_0 = \varepsilon_{\alpha_1 \cdots \alpha_s} \Omega_{\alpha_1\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{s-1}\alpha_s}$, where $\Omega_{\alpha\beta} = \sum_{i=1}^{k} \pi_{\alpha i} \wedge \pi_{i\beta}$. But the integral on the right-hand side is $\frac{1}{2}\chi(N^c)$, the Euler class of the complementary (to $\bar{N}$) s-plane bundle evaluated on the fundamental class of M. We note that this integral is independent of the choice for v as long as v is contained in $\bar{N}$. That the integral on the righthand side is the Euler class may be found in [2] and [3].

Thus we have proven:

Theorem 5. *For s odd*

$$-\frac{1}{0_{n-1}}\int_{p\times W(M)} d0_{n-1} - \frac{1}{0_s}\int_M k_v(N)dV = (-1)^{n-1}I(g,p)\ ,$$

and for s even

$$-\frac{1}{0_{n-1}}\int_{p\times W(M)} d0_{n-1} + \frac{1}{2}\chi(N^c) = (-1)^{n-1}I(g,p)\ .$$

Let us look at some specific examples. The first will be an immersion $f\colon M^s \to E^{s+2}$ of an s dimensional manifold into $(s+2)$-space where s is odd. Here $k = 2$, so we must look for a two-plane bundle, the most natural choice being the normal bundle. Thus, suppose there exists a normal vector field v on M. Then our result gives us

$$-\frac{1}{0_{s+1}}\int_{p\times W(M)} d0_{s+1} - \frac{1}{0_s}\int_M k_v(N)dV = I(g,p)\ ,$$

where $W(M)$ is the $(s+1)$-manifold swept out by the lines orthogonal to v, and $k_v(N)dV$ is the curvature form with respect to this vector field v and 2-plane normal bundle. $I(g,p)$ is the algebraic number of intersections of the normal half-planes with p.

In a very real sense $k_v(N)$ may be thought of as a generalization of the curvature of a space curve in E^3, if we choose for v the mean curvature vector of M (which we must assume now never vanishes), for we recall that kds arose in § 4 from considering the case $s = 1$ and v along the principal normal.

As the next example we choose to look at an imbedding $f\colon M^s \to E^{2s+1}$. Here, $k = s+1$, so we must look for an $(s+1)$-plane bundle. As in our prior example the most obvious is the normal bundle, but in this case we choose another. Let v be a normal vector field on M (which always exists). For $\bar{N}$ we choose the $(s+1)$-plane bundle spanned by the tangent plane and the normal vector field.

Then our result gives us

$$-\frac{1}{0_{2s}}\int_{p\times W(M)} d0_{2s} - \frac{1}{0_s}\int_M \tau_v dV = I(g,p)\ ,$$

where $W(M)$ is the tangent bundle of M, and $\tau_v dV$ is the torsion form of the imbedded manifold with respect to the vector field v [4].

What this equation gives us is a new geometric interpretation of the torsion forms introduced in [4]. We observe again that this form generalizes the torsion of a space curve if we choose v along the mean curvature vector.

6. An extension

We return now to our main theory for the following simple extension. Let us choose for N in our main equation (E) the entire $\bar{N}$ of § 5, i.e., the entire "compactified" k-plane bundle. For g we choose the map h mentioned above. Observe now that ∂N is simply the portion at infinity which we denote $W_\infty(M)$ again. Equation (E) yields

$$\frac{1}{0_{n-1}} \int_{p \times W_\infty(M)} d0_{n-1} = (-1)^{n-1} I(g, p) ,$$

where p is a point of E^n. Proceeding as before, we find

$$\frac{1}{0_{n-1}} \int_{p \times W_\infty(M)} d0_{n-1} = \frac{1}{0_{n-1}} \int_M \int_F \omega_{12} \wedge \cdots \wedge \omega_{1n} ,$$

where now F is the full $(k-1)$-sphere of points at infinity. Integration yields for s odd

$$\frac{1}{0_{n-1}} \int_M \int_F \omega_{12} \wedge \cdots \wedge \omega_{1n} = 0 ,$$

and for s even,

$$\frac{1}{0_{n-1}} \int_M \int_F \omega_{12} \wedge \cdots \wedge \omega_{1n} = \chi(N^c) ,$$

where $\chi(N^c)$ is the Euler class of the s-plane bundle complementary (to N) evaluated on the fundamental class of M.

Theorem 6. *Let $f: M^s \to E^{n=s+k}$ be an immersion of an oriented compact manifold M^s into Euclidean n-space. Let N be a k-plane bundle over M and p be a point of E^{s+k}. Then the algebraic number of intersections of the k-planes of N with p is*

(1) 0 *if s is odd,*

(2) $\chi(N^c)$ *if s is even.*

Corollary 7. *If $k = s$, the Euler characteristic of the normal bundle is the algebraic number of cross tangent planes. The Euler characteristic of M^s is the algebraic number of cross normal planes.*

Corollary 8. *If $s = 2$ and $k = 1$, the algebraic number of cross-normals is the Euler characteristic of M^2. Hence, the number of cross-normals must be even.*

(Alan Weinstein has pointed out that Corollary 8 follows from elementary Morse theory.)

7. The Gauss-Bonnet Theorem

In this section we extend the ideas of § 5 to manifolds with boundary. As might be expected, one result is a different manner of formulating the Gauss-Bonnet theorem. Let $f\colon M^2 \to E^3$ be an immersion of a surface with boundary C. As in § 1, we set $N = M^2 \times L$, $L = [-\infty, \infty]$. Let v be the unit normal vector field on M. Then we define

$$g(n) = g(m, l) = f(m) + lv_{f(m)} .$$

Equation (E) yields

$$\frac{1}{4\pi}\int_{p\times\partial N} d0_2 = I(g, p) .$$

But $\partial N = C \times L \cup M \times \{-\infty, +\infty\}$. Hence

$$-\frac{1}{4\pi}\int_{p\times[C\times L]} d0_2 - \frac{1}{4\pi}\int_{p\times M_{-\infty}} d0_2 + \frac{1}{4\pi}\int_{p\times M_{\infty}} d0_2 = I(g, p) .$$

Reasoning as in the above sections, we obtain

$$\text{(A)} \qquad -\frac{1}{4\pi}\int_{p\times[C\times L]} d0_2 + \frac{1}{2\pi}\int_M KdA = I(g, p) ,$$

where K is the Gauss curvature of M.

If we choose two different points p_1 and p_2, we have that

$$-\frac{1}{4\pi}\int_{p_i\times(C\times L)} d0_2 + \frac{1}{2\pi}\int_M KdA = I(g, p_i) , \qquad i = 1, 2 ,$$

and

$$-\frac{1}{4\pi}\int_{p_1\times(C\times L)} d0_2 + \frac{1}{4\pi}\int_{p_2\times(C\times L)} d0_2 = I(g, p_1) - I(g_1 p_2)$$

is an integer.

Let us next look at this problem with the methods of § 4. Let $f|_C\colon C \to E^3$ be the restriction of f to C. For v_1 and v_2 in § 2 we choose the normal to the surface M along the curve C and the outward pointing tangent normal along C. Then the results of § 4 yield

(B) $$-\frac{1}{4\pi}\int_{p\times L(C)} d0_2 - \frac{1}{2\pi}\int_C k_g ds = I(g', p) ,$$

where k_g is the geodesic curvature of C, $g'(L(C))$ is the surface swept out by the normal lines, and $I(g', p)$ is the algebraic number of cross-half planes at p (the half-planes being spanned by the full surface normal at C and the half outward pointing tangent normal).

But $\int_{p\times L(C)} d0_2$ is the same as $\int_{p\times(C\times L)} d0_2$, since $C \times L$ and $L(C)$ map to the same image, i.e., they are the same surface. Subtracting equation (B) from equation (A), we obtain that

$$\frac{1}{2\pi}\int_M KdA + \frac{1}{2\pi}\int_C k_g ds = I(g, p) - I(g', p)$$

is an integer. This gives us then a new way of approaching the Gauss-Bonnet formula.

The general theorem

Let f: $M^s \to E^{n=s+k}$ be an immersion of an s-dimensional oriented compact manifold with smooth boundary B^{s-1}. We divide (as for the case $s = 2$, $n = 3$) the problem into two parts. Let N be the "compactified" normal k-plane bundle over M^s. Then ∂N will consist of two parts, the portion at infinity denoted, as before, by $W_\infty(M)$ plus the restriction of the normal bundle of M to B, which we shall denote N_B. This latter part is seen to be the generalization of $C \times L$ for the case $s = 2$, $k = 1$. For the map g we take the usual map defined in previous sections. So we consider a point $p \in E^n$ such that $g(\partial N)$ does not intersect p and we apply equation (E) to obtain

$$-\frac{1}{0_{n-1}}\int_{p\times N_B} d0_{n-1} + \frac{1}{0_{n-1}}\int_{p\times W_\infty(M)} d0_{n-1} = (-1)^{n-1}I(g, p) ,$$

so that $I(g, p)$ is the sum of the indices of the intersections of the normal bundle of M with p.

The second integral is computed in precisely the same manner as in § 6 to be zero when s is odd, for the fiber F is the full sphere at infinity and thus the integration over the fiber yields zero. Hence, we obtained that

(A1) $$-\frac{1}{0_{n-1}}\int_{p\times N_B} d0_{n-1} = (-1)^{n-1}I(g, p)$$

is an integer.

However, for s even

$$\frac{1}{0_{n-1}}\int_{p\times W_\infty(M)} d0_{n-1} = \frac{1}{0_s}\int_{M^s} KdV ,$$

where KdV denotes the Gauss-Kronecker form of §§ 5 and 6, i.e.,

$$\frac{1}{0_s}\int_{M^s} KdV = \frac{(-1)^{s/2}}{2^s\pi^{s/2}(s/2)!}\int_M \Delta_0 ,$$

where Δ_0 is the form defined in § 5. Note that since we are integrating over the full sphere F the expression in § 5 is multiplied by a factor of 2.

Thus, we have

$$\text{(A2)} \qquad -\frac{1}{0_{n-1}}\int_{p\times N_B} d0_{n-1} + \frac{1}{0_s}\int_{M^s} KdV = (-1)^{n-1}I(g,p) .$$

The second part proceeds as follows. We consider just the restriction of f to B^{s-1} and use the methods of § 5.

For the vector field v we choose the uniquely defined tangent normal pointing outward (tangent to M^s and normal to B^{s-1}). Next we determine the half-$(k+1)$-plane bundle N'. Let $\bar{N}'$ denote the normal bundle of B^{s-1}. Then each half-plane, say at $f(b)$, will be that determined by the full k-plane orthogonal to $v_{f(b)}$ and the positive or outward pointing $v_{f(b)}$. Then ∂N consists of the portion at infinity which we denote $W'_\infty(B)$ plus the k-plane bundle to B^{s-1}, which is easily seen to be the N_B of the first part. The map g' will again be the usual map. We apply equation (E) and obtain

$$-\frac{1}{0_{n-1}}\int_{N_B} d0_{n-1} + \frac{1}{0_{n-1}}\int_{p\times W'_\infty(B)} d0_{n-1} = (-1)^{n-1}I(g',p) ,$$

where we use, of course, the same point as before.

We integrate the second integral as before to obtain for s odd

$$\frac{1}{0_{n-1}}\int_{p\times W'_\infty(B)} d0_{n-1} = \frac{1}{2}\chi(N'^c) ,$$

where $\chi(N'^c)$ is the Euler class of the $(s-1)$-plane bundle complementary (to N') evaluated on the fundamental class of B. But this $(s-1)$-plane bundle is just the tangent bundle of B^{s-1}. Hence

$$\chi(N'^c) = \chi(B^{s-1}) ,$$

where $\chi(B^{s-1})$ is the Euler characteristic of B^{s-1}.

For s even, we obtain

$$\frac{1}{0_{n-1}}\int_{p\times W'_\infty(B)} d0_{n-1} = -\frac{1}{0_{s-1}}\int_B k_v(N')dV_B ,$$

where the expressions for $k_v(N')dV_B$ are given in § 5. Then subscript B distinguishes dV_B from dV above in KdV. Summing up, we have for s odd

$$\text{(B1)} \qquad -\frac{1}{0_{n-1}}\int_{N_B} d0_{n-1} + \frac{1}{2}\chi(B) = (-1)^{n-1}I(g', p) ,$$

and for n even

$$\text{(B2)} \qquad -\frac{1}{0_{n-1}}\int_{N_B} d0_{n-1} - \frac{1}{0_{s-1}}\int_B k_v(N')dV_B = (-1)^{n-1}I(g', p) .$$

If we subtract (B1) and (B2) from (A1) and (A2) respectively, we obtain for s odd

$$-\frac{1}{2}\chi(B^{s-1}) = (-1)^{n-1}[I(g, p) - I(g', p)] ,$$

and for s even

$$\frac{1}{0_s}\int_{M^s} KdV + \frac{1}{0_{s-1}}\int_{B^{s-1}} k_v(N')dV_B = (-1)^{n-1}[I(g, p) - I(g', p)] .$$

The result for s odd gives the well known theorem that the Euler characteristic of an even dimensional manifold which bounds is even.

The result for s even gives the new approach to the generalized Gauss-Bonnet theorem. We observe that the forms $k_v(N')dV_B$ are, of course, the forms introduced by Chern to prove the Gauss-Bonnet formula for closed manifolds in [1].

8. On the genericity of p

In closing, we mention a manner in which p need not be assumed generic. For example, suppose $p \in N$ (perhaps even $\in \partial N$), say in § 3, $p \in M$. This case may be dealt with using the techniques of this paper together with those of [4], i.e., we use the space of secants $S(p, N)$ of N relative to the point p. For details refer to [4].

Added in proof. Professor Carl B. Allendoerfer has pointed out that the curvature form invariants of § 5 have previously been used for purposes other than this paper by C. B. Allendoerfer (*Characteristic cohomology classes in* *a Riemann manifold*, Ann. of Math. **51** (1950) 551–570), by A. Aragnol (*Classes caractéristiques et formes différentielles*, C. R. Acad. Sci. Paris **238** (1954) 2387–2389), and by J. Eells, Jr. (*A generalization of the Gauss-Bonnet theorem*, Trans. Amer. Math. Soc. **92** (1959) 142–153). What this present paper gives is entirely new interpretations for these invariants. The author wishes to thank Professor Allendoerfer for many helpful criticisms of this paper.

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